Outermost boundaries for star-connected components in percolation

Ghurumuruhan Ganesan *

NISER, Bhubaneshwar, India

Abstract

Tile $\mathbb{R}^2$ into disjoint unit squares $\{S_k\}_{k\geq 0}$ with the origin being the centre of S_0 and say that S_i and S_j are star-adjacent if they share a corner and plus-adjacent if they share an edge. Every square is either vacant or occupied. If the occupied plus-connected component $C^+(0)$ containing the origin is finite, it is known that the outermost boundary ∂_0^+ of $C^+(0)$ is a unique cycle surrounding the origin. For the finite occupied star-connected component $C(0)$ containing the origin, we prove in this paper that the outermost boundary ∂_0 is a unique connected graph consisting of a union of cycles $\cup_{1\leq i\leq n}C_i$ with mutually disjoint interiors. Moreover, we have that each pair of cycles in ∂_0 share at most one vertex in common and we provide an inductive procedure to obtain a circuit containing all the edges of $\cup_{1\leq i\leq n}C_i$. This has applications for contour analysis of star-connected components in percolation.

Key words: Star connected components, outermost boundary, union of cycles.

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*E-Mail: gganesan82@gmail.com

1 Introduction

Tile $\mathbb{R}^2$ into disjoint unit squares $\{S_k\}_{k\geq 0}$ with origin being the centre of S_0. We say S_1 and S_2 are *adjacent* or *star-adjacent* if they share a corner between them. We say that squares S_1 and S_2 are *plus-adjacent*, if they share an edge between them. Here we follow the notation of Penrose (2003). Suppose every square is assigned one of the two states: occupied or vacant. In many applications like for example, percolation, it is of interest to determine the outermost boundary of the plus-connected or star-connected components containing the origin. We make formal definitions below. The case of plus-connected components is well studied (Bollobas and Riordan (2006), Penrose (2003)) and in this case, the outermost boundary is simply a cycle containing the origin. Our main result is that the outermost boundary for the star-connected component is a connected union of cycles with disjoint interiors.

Let $C(0)$ denote the star-connected occupied component containing the origin and throughout we assume that $C(0)$ is finite. Thus if S_0 is vacant then $C(0) = \emptyset$. Else $S_0 \in C(0)$ and if $S_1, S_2 \in C(0)$ there exists a sequence of distinct occupied squares $(Y_1, Y_2, ..., Y_t)$ all belonging to $C(0)$, such that Y_i is adjacent to Y_{i+1} for all i and $Y_1 = S_1$ and $Y_t = S_2$. Let G_C be the graph with vertex set being the set of all corners of the squares $\{S_k\}_k$ in $C(0)$ and edge set consisting of the edges of the squares $\{S_k\}_k$ in $C(0)$.

Two vertices u and v are said to be adjacent in G_C if they share an edge between them. We say that an edge e in G_C is adjacent to square S_k if it is one of the edges of S_k. We say that e is a *boundary edge* if it is adjacent to a vacant square and is also adjacent to an occupied square. A *path* P in G_C is a sequence of distinct vertices $(u_0, u_1, ..., u_t)$ such that u_i and u_{i+1} are adjacent for every i. A *cycle* C in G_C is a sequence of distinct vertices $(v_0, v_1, ..., v_m, v_0)$ starting and ending at the same point such that v_i is adjacent to v_{i+1} for all $0 \leq i \leq m-1$ and v_m is adjacent to v_0. A *circuit* C' in G_C is a sequence of vertices $(w_0, w_1, ..., w_r, w_0)$ starting and ending at the same point such that w_i is adjacent to w_{i+1} for all $0 \leq i \leq r-1$, w_r is adjacent to w_0 and no edge is repeated in C'. Thus vertices may be repeated in circuits and for more related definitions, we refer to Chapter 1, Bollobas (2001).

Any cycle C divides the plane $\mathbb{R}^2$ into two disjoint connected regions. As in Bollobas and Riordan (2006), we denote the bounded region to be the *interior* of C and the unbounded region to be the *exterior* of C. We have the following definition.

Definition 1. *We say that edge e in G_C is an* outermost boundary *edge of the component $C(0)$ if the following holds true for every cycle C in G_C : either e is an edge in C or e belongs to the exterior of C.*

We define the outermost boundary ∂_0 of $C(0)$ to be the set of all outermost boundary edges of G_C.

Thus outermost boundary edges cannot be contained in the interior of any cycle in G_C. Our main result is the following.

Theorem 1. *Suppose $C(0)$ is finite. The outermost boundary ∂_0 of $C(0)$ is a unique set of cycles $C_1, C_2, ..., C_n$ in G_C with the following properties:*
(i) The graph $\cup_{1 \leq i \leq n} C_i$ is a connected subgraph of G_C.
(ii) If $i \neq j$, the cycles C_i and C_j have disjoint interiors and share at most one vertex.
(iii) Every square $S_k \in C(0)$ is contained in the interior of some cycle C_j.
(iv) If $e \in C_j$ for some j, then e is a boundary edge of $C(0)$ adjacent to an occupied square of $C(0)$ in the interior of C_j and also adjacent to a vacant square in the exterior.

Moreover, there exists a circuit C_{out} containing every edge of $\cup_{1 \leq i \leq n} C_i$.

The outermost boundary ∂_0 is therefore also an Eulerian graph with C_{out} denoting the corresponding Eulerian circuit (for definitions, we refer to Chapter 1, Bollobas (2001)). We remark that the above result also provides a more detailed justification of the statement made about the outermost boundary and the corresponding circuit in the proof of Lemma 3 of Ganesan (2013). Using the above result, we also obtain the outermost circuit that is used to construct the top-down crossing in oriented percolation in a rectangle in Ganesan (2015).

The proof of the above result also obtains the outermost boundary cycle in the case of plus-connected components. We recall that S_1 and S_2 are *plus-adjacent* if they share an edge between them. Analogous to the star-connected case, we define $C^+(0)$ to be the plus-connected component containing the origin and define the graph G_C^+ consisting of edges and corners of squares in $C^+(0)$. We have the following.

Theorem 2. *Suppose $C^+(0)$ is finite. The outermost boundary ∂_0^+ of $C^+(0)$ is unique cycle C_{out}^+ in G_C^+ with the following property:*
(i) All squares of $C^+(0)$ are contained in the interior of C_{out}^+.
(ii) Every edge in C_{out}^+ is a boundary edge adjacent to an occupied square of $C^+(0)$ in the interior of C_{out}^+ and a vacant square in the exterior.

This is in contrast to star-connected components which may contain multiple cycles in the outermost boundary.

To prove Theorem 1, we use the following intuitive result about merging cycles. Analogous to G_C, let G be the graph with vertex set being the corners of the squares $\{S_k\}_k$ and edge set being the edges of the squares $\{S_k\}_k$.

Theorem 3. *Let C_1 and C_2 be cycles in G that have more than one vertex in common. There exists a unique cycle C_3 consisting only of edges of C_1 and C_2 with the following properties:*
(i) the interior of C_3 contains the interior of both C_1 and C_2,
(ii) if an edge e belongs to C_1 or C_2, then either e belongs to C_3 or is contained in its interior.

Moreover, if C_2 contains at least one edge in the exterior of C_1, then the cycle C_3 also contains an edge of C_2 that lies in the exterior of C_1.

The above result essentially says that if two cycles intersect at more than one point, there is a innermost cycle containing both of them in its interior. We provide an iterative construction for obtaining the cycle C_3, analogous to Kesten (1980) for crossings, in Section 3.

The paper is organized as follows: In Section 2, we prove Theorem 1 and in Section 3, we prove Theorem 2 and Theorem 3.

2 Proof of Theorem 1

Proof of Theorem 1: The first step is to obtain large cycles surrounding each occupied square in $C(0)$. We have the following Lemma.

Lemma 4. *For every $S_k \in C(0)$, there exists a unique cycle D_k satisfying the following properties:*
(a) S_k is contained in the interior of D_k,
(b) every edge in the cycle D_k is a boundary edge adjacent to one occupied square of $C(0)$ in the interior and one vacant square in the exterior and
(c) if C is any cycle in G_C that contains S_k in the interior, then every edge in C either belongs to D_k or is contained in the interior.

We denote D_k to be the outermost boundary cycle containing the square S_k. We prove all statements at the end.

We claim that the set of distinct cycles in the set $\mathcal{D} := \cup_{S_k \in C(0)}\{D_k\}$ is the desired outermost boundary ∂_0 and satisfies the conditions (i)-(iv)

mentioned in the statement of the theorem. By construction, we have that (iii) and (iv) are satisfied. To see that (ii) holds, we suppose that $D_{k_1} \neq D_{k_2}$ and that D_{k_1} and D_{k_2} meet at more than one vertex. We know that D_{k_2} is not completely contained in D_{k_1}. Thus D_{k_2} contains at least one edge in the exterior of D_{k_1}. From Theorem 3, we obtain a cycle D'_{12} containing both D_{k_1} and D_{k_2} in the interior and containing an edge e present in D_{k_2} but not in D_{k_1} or its interior. The cycle D'_{12} satisfies condition (a) in Lemma 4 above and thus contradicts the assumption that D_{k_1} satisfies (c). Thus D_{k_1} and D_{k_2} cannot meet at more than one vertex.

Also (i) holds, because of the following reason. First we note that by construction G_C is connected; let u_1 and u_2 be vertices in G_C. Each $u_i, i = 1, 2$ is a corner of an occupied square $S_i \in C(0)$ and by definition, S_1 and S_2 are star-connected via squares in $C(0)$. Thus there exists a path in G_C from u_1 to u_2.

To see that $\mathcal{D}$ is a connected subgraph of G_C, we let v_1 and v_2 be vertices in $\mathcal{D}$ that belong to cycles D_{r_1} and D_{r_2}, respectively, for some r_1 and r_2. If $r_1 = r_2$, then v_1 and v_2 are connected by a path in $D_{r_1} = (z_1 = v_1, z_2, ..., z_n, z_1)$. If $r_1 \neq r_2$, let $P_{12} = (w_1 = v_1, w_2, ..., w_{t-1}, w_t = v_2)$ be a path from v_1 to v_2 in G_C. We iteratively construct a path P'_{12} from P_{12} using only edges of cycles in $\mathcal{D}$. We first note that since (iii) holds, every edge in P_{12} either belongs to a cycle in $\mathcal{D}$ or is contained in the interior of some cycle in $\mathcal{D}$. Let i_1 be the first time P_{12} leaves D_{r_1}; i.e., let $i_1 = \min\{i \geq 1 : w_{i+1}$ belongs to exterior of $D_{r_1}\}$.

The edge formed by the vertices w_{i_1} and w_{i_1+1} belongs to some cycle $D_{s_1} = (x_1 = w_{i_1}, x_2, ..., x_r, x_1)$ or is contained in its interior. Since the cycles D_{r_1} and D_{s_1} have disjoint interiors, this necessarily means D_{s_1} and D_{r_1} meet at w_{i_1}. Defining $T_1 = (z_1 = v_1, z_2, ..., z_{j_1} = w_{i_1})$, we note that T_1 is a path consisting only of edges in the cycle D_{r_1} and containing the vertex $z_1 = v_1$. Repeating the same procedure above, we obtain another path $T_2 = (w_{i_1} = x_1, x_2, ..., x_{j_2} = w_{i_2})$ contained in D_{s_1}, where, as before, $i_2 = \min\{i \geq i_1 + 1 : w_{i+1}$ belongs to exterior of $D_{s_1}\}$ denotes the first time P_{12} leaves D_{s_1}. We continue this procedure for a finite number of steps m, until we reach v_2. By construction, the path T_i obtained at step $i, 2 \leq i \leq m$ is connected to $\cup_{1 \leq j \leq i-1} T_j$. The final union of paths $\cup_{1 \leq i \leq m} T_i$ is therefore a connected graph containing only edges in $\mathcal{D}$ and contains v_1 and v_2.

It remains to see that an edge e belongs to the outermost boundary if and only if it belongs to some cycle in $\mathcal{D}$. If e is an edge in a cycle $D_k \in \mathcal{D}$ we have that e is adjacent to an occupied square S_e contained in the interior

of D_k and a vacant square S'_e in the exterior. If there exists a cycle C in G_C that contains e in the interior, we then have that both S_e and S'_e are contained in the interior of C. Since S'_e is exterior to D_k, the cycle C contains at least one edge in the exterior of D_k. But if D_e denotes the outermost cycle containing S_e, then by the discussion in the first paragraph, we must have that $D_e = D_k$. And thus every edge of C either belongs to D_e or is contained in the interior of D_e which leads to a contradiction.

We also see that no other edge apart from edges of cycles in $\mathcal{D}$ can belong to the outermost boundary since if $e_1 \notin \mathcal{D}$, then e_1 is necessarily contained in the interior of some cycle $D_r \in \mathcal{D}$.

Finally, to obtain the circuit we compute the cycle graph H_{cyc} as follows: let $E_1, E_2, ..., E_n$ be the distinct outermost boundary cycles in $\mathcal{D}$. Represent E_i by a vertex i in H_{cyc}. If E_i and E_j share a corner, we draw an edge between i and j. We have the following lemma.

Lemma 5. *We have that the graph H_{cyc} described above is a tree.*

We provide the proof of the above at the end.

We then obtain the circuit via induction on the number of vertices n of H_{cyc}. For $n = 1$, it is a single cycle. Suppose we obtain the circuit of all cycle graphs containing at most k vertices and let H_{cyc} be a cycle graph containing $k+1$ vertices. To obtain the circuit for H_{cyc}, we pick a leaf q of H_{cyc} and apply induction assumption on the cycle graph $H'_{cyc} = H_{cyc} \setminus q$. To fix a procedure, we choose q such that the corresponding boundary cycle E_q contains a square S_j of least index j in its interior. We have that H'_{cyc} is connected and has k vertices and thus has a circuit $C_k = (c_1, c_2, ..., c_r, c_1)$ containing all edges of every cycle in H'_{cyc}. Let C_k meet the cycle $E_q = (d_1, d_2, ..., d_i, d_1)$ at $d_t = c_1$. We then form the new circuit $C_{k+1} = (d_1, d_2, ..., d_t = c_1, c_2, ..., c_r, c_1 = d_t, d_{t+1}, ..., d_i, d_1)$, which contains all edges of every cycle in H_{cyc}. ■

Proof of Lemma 4: We note that if there exists such a D_k, then it is unique by definition. Let $\mathcal{E}$ be the set of all cycles in G_C satisfying condition (a); i.e., if C is a cycle containing S_k in its interior then $C \in \mathcal{E}$. The set $\mathcal{E}$ is not empty since the cycle formed by the four edges of S_k belongs to $\mathcal{E}$. We merge cycles in $\mathcal{E}$ two by two using Theorem 3 to obtain the desired cycle D_k. We first pick a cycle F_1 in $\mathcal{E}$ using a fixed procedure; for example, using an analogous iterative procedure as described in Section 1 of Ganesan (2014) for choosing paths.

We again use the same procedure to pick a cycle F_2 in $\mathcal{E} \setminus F_1$ and from Theorem 3, obtain a cycle F_1' consisting of only edges of F_1 and F_2 and containing both F_1 and F_2 in its interior. The cycle F_1' also satisfies (a) and thus belongs to $\mathcal{E}$. Therefore, if $\mathcal{E}$ has t cycles, then $\mathcal{E}_1 := (\mathcal{E} \setminus \{F_1, F_2\}) \cup F_1'$ has at most $t-1$ cycles; if F_1 contains an edge in the exterior of F_2 and the cycle F_2 also contains an edge in the exterior of F_1, then $\mathcal{E}_1$ has $t-2$ cycles. Else F_1' is either F_1 or F_2 and the set $\mathcal{E}_1$ therefore contains $t-1$ cycles.

By construction, every cycle in $\mathcal{E}$ is either a cycle in $\mathcal{E}_1$ or is contained in the interior of a cycle in $\mathcal{E}_1$. Therefore, if $\mathcal{E}_1$ contains one cycle, it is the desired outermost boundary cycle D_k. Else we repeat the above procedure with $\mathcal{E}_1$ and obtain another set $\mathcal{E}_2$ containing at most $t-2$ cycles and again with the property that every cycle in $\mathcal{E}$ is either a cycle in $\mathcal{E}_2$ or is contained in the interior of a cycle in $\mathcal{E}_2$. Continuing this process, we are finally left with a single cycle C_{fin}. By construction it satisfies (a) and (c). It only remains to see that (b) is true.

Suppose there exists an edge e of C_{fin} that is not a boundary edge. Since e is an edge of G_C, we then have that e is adjacent to two occupied squares S_1 and S_2, with one of the squares, say S_1, contained in the interior of C_{fin} and the other square S_2, contained in the exterior. The cycle C_2 containing the four edges of the square S_2 and the cycle C_{fin} have the edge e in common and thus more than one vertex in common. Since C_2 contains at least one edge in the exterior of C_{fin}, we use Theorem 3 to obtain a larger cycle C_2' containing both C_{fin} and C_2 in the interior. The cycle C_2' contains at least one edge not in C_{fin}. But since C_{fin} satisfies (c), this is a contradiction. Thus every edge e of C_{fin} is a boundary edge.

By the same argument above, we also see that the edge e cannot be adjacent to an occupied square in the exterior of C_{fin}. Thus e is adjacent to an occupied square in the interior and a vacant square in the exterior. ■

Proof of Lemma 5: We already have that H_{cyc} is connected. It is enough to see that it is acyclic. Before we prove that, we make the following observation. Consider a path $P = (i_1, i_2, ..., i_m)$ in H_{cyc}. We see that any vertex in E_{i_1} and any vertex in E_{i_m} is connected by a path consisting only of edges of the cycles $\{E_{i_k}\}_{1 \leq k \leq m}$.

Suppose H_{cyc} contains a cycle $C = (r_1, r_2, ..., r_s, r_1)$. Let the boundary cycle $E_{r_1} = (u_1, u_2, ..., u_m, u_1)$ meet E_{r_2} at u_1 and E_{r_s} at u_j. We have that $j \neq 1$ since three boundary cycles cannot meet at a point. This is illustrated in Figure 1(b). The occupied square S_1 belongs to E_{r_2} and the occupied

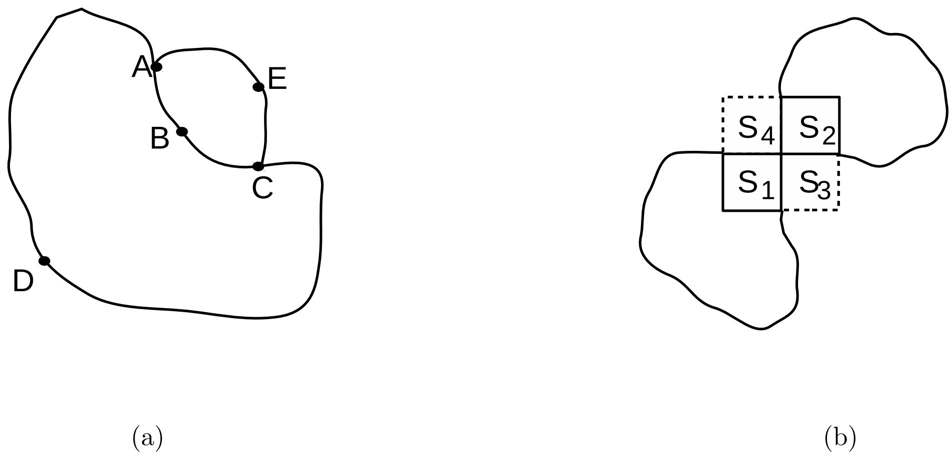

Figure 1: (a) Merging cycle $ABCDA$ with the segment AEC. (b) Only two cycles can meet at a single point.

square S_2 belongs to E_{r_s}. It is necessary that the squares S_3 and S_4 are vacant and thus cannot be on the boundary of any other cycle.

Let P_1 and P_1' be the two segments of E_{r_1} starting at u_1 and ending at u_j. Since $u_1 \in E_{r_2}$ and $u_j \in E_{r_s}$, we have by the observation made in the first paragraph that there exists a path P_2 from u_1 to u_j, consisting only of edges in $\{E_{r_i}\}_{2 \leq i \leq s}$. This path necessarily lies in the exterior of E_{r_1} and is illustrated in Figure 1(a). Here $ABCDA$ represents the cycle E_{r_1}, the path P_1 is the segment ADC and the path P_1' is the segment ABC. The path P_2 is denoted by the exterior segment AEC.

Thus it is necessary that either the cycle C_{12} formed by $P_1 \cup P_2$ contains P_1' in the interior or the cycle C_{12}' formed by $P_1' \cup P_2$ contains P_1 in the interior. Suppose the former holds and let S_a be any occupied square in the interior of E_{r_1}. We know that $E_{r_1} = D_a$ is the outermost boundary cycle containing S_a and satisfies conditions (a), (b) and (c) mentioned in Lemma 4. The cycle C_{12} also contains S_a in the interior and thus satisfies condition (a). Moreover, it contains at least one edge in the exterior of E_{r_1} contradicting the fact that E_{r_1} satisfies (c). Thus H_{cyc} is acyclic. ∎

3 Proofs of Theorem 2 and Theorem 3

Proof of Theorem 2: Let D_0 be the outermost boundary cycle containing the square S_0 as in Lemma 4. It satisfies the conditions (i) and (ii) in the statement of the theorem and is unique and thus $C^+_{out} = D_0$. ∎

Proof of Theorem 3: If every edge of C_1 is either on C_2 or contained in the interior of C_2, then the desired cycle $C_3 = C_2$. If similarly, C_2 is completely contained in C_1, we set $C_3 = C_1$. So we suppose that C_1 contains at least one edge in the exterior of C_2 and C_2 also contains at least one edge in the exterior of C_1.

We start with cycle C_1 and in the first step, identify a path of C_2 contained in the exterior of C_1. Set $C_{1,0} := C_1 = (u_0, u_1, ..., u_{t-1}, u_0)$ and $C_2 = (v_0, v_1, ..., v_{m-1}, v_0)$. For later notation, we define $u_k = u_{k \mod t}$ if $k \leq 0$ or $k \geq t$ and $v_k = v_{k \mod m}$ if $k \leq 0$ or $k \geq m$.

Start from some vertex of $C_{1,0}$, say u_0, and look for the first intersection point that contains an exterior edge of C_2; i.e., an edge of C_2 that lies in the exterior of C_1. Let

$$j_1 = \min\{j \geq 0 : u_j \in C_{1,0} \text{ and } u_j \text{ is an endvertex of an exterior edge of } C_2\}$$

and let $v_{i_1} = u_{j_1}$. We suppose that the edge of C_2 with endvertices v_{i_1} and v_{i_1+1} lies in the exterior of C_1. Let $r_1 = \min\{i \geq i_1 + 1 : v_i \in C_{1,0}\}$ be the next time the cycles meet and define $P_1 = (v_{i_1}, v_{i_1+1}, ..., v_{r_1})$.

We note that none of the vertices $v_j, i_1 + 1 \leq j \leq r_1 - 1$ belong to $C_{1,0}$. If $v_{i_1} = v_{r_1}$, then P_1 is a cycle containing the edges of C_2 and thus $P_1 = C_2$. Since C_1 and C_2 contain more than one vertex in common, this cannot happen. Thus P_1 is a path and all edges of P_1 are in the exterior of $C_{1,0}$.

We then construct an outermost cycle from $C_{1,0}$ and P_1 as follows. Split $C_{1,0}$ into two segments based on intersection with P_1. Suppose P_1 meets $C_{1,0}$ at u_{a_1} and u_{b_1}. We let $C'_{1,0} = (u_{a_1}, u_{a_1+1}, ..., u_{b_1})$ and $C''_{1,0} = (u_{a_1}, u_{a_1-1}, ..., u_{b_1})$. If the interior of $C'_{1,0} \cup P_1$ contains the interior of $C''_{1,0} \cup P_1$ as in Figure 1(a), we set $C_{1,1} = C'_{1,0} \cup P_1$ to be the cycle obtained in the first iteration by the concatenation of the paths $C'_{1,0}$ and P_1. Here $C''_{1,0}$ is the segment ADC, the path $C'_{1,0}$ is the segment ABC and the path P_1 is denoted AEC. Else necessarily we have that the interior of $C''_{1,0} \cup P_1$ contains the interior of $C'_{1,0} \cup P_1$ and we set $C_{1,1} = C''_{1,0} \cup P_1$. Since $P_1 \neq \emptyset$, we have that $C_{1,1}$ contains at least one exterior edge.

We then perform the same procedure as above on the cycle $C_{1,1}$ and continue this process for a finite number of steps to obtain the final cycle $C_{1,n}$. For each $j, 1 \leq j \leq n$, we have that the cycle $C_{1,j}$ satisfies the following properties:
(1) the cycle $C_{1,j}$ contains only edges from C_1 and C_2,
(2) every edge of C_1 either belongs to $C_{1,j}$ or is contained in the interior of $C_{1,j}$,
(3) the cycle $C_{1,j}$ contains at least one exterior edge of C_2 and
(4) the interior of C_1 is contained in $C_{1,j}$.
In particular, the above properties hold true for the final cycle $C_{1,n}$. If there exists an edge e of C_2 in the exterior of $C_{1,n}$, then the edge e belongs to a path P_e of C_2 containing edges exterior to C_1. The path P_e must meet C_1 and thus there exists an edge of C_2 that lies in the exterior of C_1 and contains an endvertex of C_1. But then the above procedure would not have terminated and thus we also have:
(5) every edge of C_2 either belongs to $C_{1,n}$ or is contained in the interior of $C_{1,n}$.

Thus property (ii) stated in the result holds true and we need to see that (i) holds. For that we first prove uniqueness of the cycle $C_{1,n}$ obtained above. Suppose there exists another cycle D' satisfying properties (1), (2) and (5) above. If D' contains an edge e' (which must necessarily belong to C_1 or C_2) in the exterior of $C_{1,n}$, it contradicts the fact that $C_{1,n}$ satisfies (2) and (5). If D' is completely contained in the interior of $C_{1,n}$ and is not equal to $C_{1,n}$, then there is at least one edge of $C_{1,n}$ (which belongs to C_1 or C_2) that lies in the exterior of D', contradicting the assumption that D' satisfies (2) and (5).

Thus any cycle satisfying properties (1), (2) and (5) is unique. We recall that C_1 also contains an edge in the exterior of C_2. Suppose now we start from $C_{2,0} := C_2$ and identify segments of C_1 lying in the exterior of C_2 and perform the same iterative procedure as above to obtain a final cycle $C_{2,m}$. This cycle must also satisfy (1), (2) and (5) and hence $C_{2,m} = C_{1,n}$. Moreover, $C_{2,m}$ satisfies:
(3′) the cycle $C_{2,m}$ contains at least one exterior edge of C_1 and
(4′) the interior of C_2 is contained in $C_{2,m}$.

Thus the cycle $C_{1,n}$ is unique and satisfies properties (i) and (ii) stated in the result. ■

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References

[1] B. Bollobas. (2001). *Modern Graph Theory.* Springer.

[2] B. Bollobas and O. Riordan. (2006). *Percolation.* Academic Press.

[3] G. Ganesan. (2013). Size of the giant component in a random geometric graph. *Ann. Inst. Henri Poincare*, **49**, 1130–1140.

[4] G. Ganesan. (2014). First passage percolation with nonidentical passage times. *Arxiv Link*: http://arxiv.org/abs/1409.2602

[5] G. Ganesan. (2015). Infection spread in random geometric graphs. *Adv. Appl. Probab.*, **47**, 164–181.

[6] H. Kesten. (1980). The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$. *Commun. Math. Phys.*, **74**, 41–59.

[7] M. Penrose. (2003). *Random Geometric Graphs.* Oxford.